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PUSHED TRAVELLING WAVES IN AN INITIAL-BOUNDARY VALUE PROBLEM FOR FISHER TYPE EQUATIONS

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Pushed travelling waves in an initial-boundary value problem for Fisher type equations *)

by

E.J.M. Veling

ABSTRACT

In this paper we consider the initial-boundary value problem for the semilinear diffusion equation $u_t = u_{xx} + f(u)$ on the half-line $x > 0$, when $f(0) = f(1) = 0$ and $f(u) > 0$ if $0 < u < 1$. For a wide class of initial and boundary values a uniformly valid asymptotic expression will be given to which the solution converges exponentially. This expression is composed of a travelling wave with the minimal possible velocity $c(f) > 2\sqrt{f'(0)}$ and a solution of the stationary problem.

KEY WORDS & PHRASES: *semilinear diffusion, initial-boundary value problem, pulled and pushed travelling wave, Lyapunov functional, exponential stability*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we shall give results about an initial-boundary value problem for the semilinear diffusion equation

$$(1.1) \qquad u_t = u_{xx} + f(u), \quad (x,t) \in Q = (\mathbb{R}^+ \times \mathbb{R}^+)$$

where f satisfies

$$\text{(Hf1)} \qquad \begin{array}{l} f \in C^1([0,1]), \quad f(0) = f(1) = 0, \quad f'(0) > 0, \\ f'(1) < 0, \quad f(u) > 0, \quad 0 < u < 1. \end{array}$$

This equation can serve as a model for the spread of an allele (A) in a diploid population with zygotes AA, Aa and aa. The function u, which depends on time t and the spatial variable x, represents the frequency of the allele A. Condition (Hf1) stands for the "heterzygote (Aa) intermediate" case, see FISHER [4] and ARONSON & WEINBERGER [1]. Fisher used in his model for the nonlinearity the expression

$$(1.2) \qquad f_F = u(1-u)(1-\tau-(2-\sigma-\tau)u)$$

where τ and σ are measures for the relative fitness of the homozygotes AA and aa with respect to the heterozygote Aa. Condition (Hf1) implies that $\sigma > 1$ and $0 < \tau < 1$. As is well-known, this type of semilinear parabolic equation allows *travelling wave* solutions, i.e. solutions of the form $u(x,t) = U(z)$, $z = x - ct$, where U satisfies the ordinary differential equation

$$(1.3) \qquad \begin{cases} U'' + cU' + f(U) = 0, \quad z \in \mathbb{R}, \\ \lim_{z\to-\infty} U(z) = 1, \quad \lim_{z\to\infty} U(z) = 0. \end{cases}$$

For functions satisfying (Hf1) there exists a critical wavespeed $c(f)$, such that for all $c \geq c(f) > 0$ there exists a travelling wave $U_c(z)$, which is unique modulo translation. Therefore we fix $U_c(z)$ by the requirement $U_c(0) = \frac{1}{2}$. $U_c(z)$ is monotonically decreasing. See for example ROTHE [8] and UCHIYAMA [11] for these results. A priori the minimal velocity $c(f)$ can be bounded by

$$(1.4) \qquad f'(0) \le c^2(f)/4 \le \max_{0\le u\le 1} f(u)/u.$$

Depending on the properties of the function f there arises now two cases which we have to distinguish, namely $c(f) = 2\sqrt{f'(0)}$ and $c(f) > 2\sqrt{f'(0)}$. The names *pulled* and *pushed waves* respectively were introduced by STOKES [10] for these two cases. These names can be explained by realizing that in a pulled wave the velocity of the wave is apparently determined only by the behaviour of the function f in $u = 0$, so the tail of U pulls the wave to the right, while for a pushed wave the velocity is also influenced by the values $f(u)$ for $u > 0$. See further UCHIYAMA [11; §§1,2].

If we consider the pure initial value problem, i.e. (1.1) together with the condition $u(x,0) = g(x)$, $x \in \mathbb{R}$, then it is known that under certain conditions on g the solution $u(x,t;g)$ of this problem converges to a travelling wave U, see KOLMOGOROFF, PETROVSKY & PISCOUNOFF [6] for the first results and also ROTHE [8], STOKES [10] and UCHIYAMA [11] among others for generalizations with respect to the class of admissible functions g. The crucial information for these results turns out to be the manner in which $g(x) \to 0$, as $x \to \infty$.

The asymptotic behaviour of the travelling wave $U_c(z)$ can easily be determined by linearizing around $U = 0$ and $U = 1$. As $z \to -\infty$, $U(z) \to 1$ as follows

$$(1.5) \qquad \begin{aligned} &1 - U_c(z) = C_1 e^{\beta_1 z}(1+o(1)), \quad z \to -\infty, \\ &\beta_1 \equiv -\tfrac{1}{2}\,[c - \sqrt{c^2-4f'(1)}] > 0, \end{aligned}$$

As $z \to \infty$, $U(z) \to 0$, but the behaviour near $U = 0$ depends on whether or not the speed of $U_c(z)$ is equal to $c(f)$:

$$(1.6) \qquad \begin{aligned} &U_c(z) = C_2 e^{\beta_2 z}(1+o(1)), \quad z \to \infty, \\ &\beta_2 \equiv -\tfrac{1}{2}[c + \sqrt{c^2-4f'(0)}] < 0, \quad \text{if } c = c(f), \end{aligned}$$

$$U_c(z) = C_3 e^{\beta_3 z}(1+o(1)), \qquad z \to \infty,$$

(1.7)

$$\beta_3 \equiv -\frac{1}{2}\,[c-\sqrt{c^2-4f'(0)}] < 0, \qquad \text{if } c > c(f).$$

See UCHIYAMA [11] for a thorough treatment of this asymptotic behaviour. It turns out that the expression $|\beta_2(c) - \beta_3(c)| = \sqrt{c^2-4f'(0)}$ is bounded away from zero for all $c \geq c(f)$ in the pushed case, while in the pulled case $|\beta_2(c(f)) - \beta_3(c(f))| = 0$.

The limit behaviour of u for the Cauchy problem can be summarized as follows. Assume $g(x) = O(e^{-\mu x})$, $x \to \infty$, then for $\mu \geq -\beta_3(c(f))$, u tends to $U_{c(f)}$ in form if $t \to \infty$, while for $0 < \mu < -\beta_3(c(f))$, u tends to U_c in form, $t \to \infty$, where $c = \mu + f'(0)/\mu$ is such that $U_c = O(e^{-\mu z})$, $z \to \infty$. Thus u picks for U_c that travelling wave, which has the same asymptotic behaviour as g at infinity. For these results we refer to ROTHE [8] and UCHIYAMA [11]. For the pushed case it can be proved that the convergence to the travelling wave with minimal speed is not only in form but even pointwise (ROTHE [9]).

We shall consider here the corresponding initial-boundary value problem for $(x,t) \in (\mathbb{R}^+ \times \mathbb{R}^+)$. So besides the initial condition $g(x)$, $x \in \overline{\mathbb{R}^+}$, we have to specify a boundary condition $u(0,t) = h(t)$, $t \in \overline{\mathbb{R}^+}$. Thus we study the problem

(P)
$$\begin{cases} u_t = u_{xx} + f(u), & (x,t) \in Q = (\mathbb{R}^+ \times \mathbb{R}^+), \\ u(x,0) = g(x), & x \in \overline{\mathbb{R}^+}, \\ u(0,t) = h(t), & t \in \overline{\mathbb{R}^+}. \end{cases}$$

In VELING [12] we considered the "heterozygote inferior" case for the nonlinearity f; it meant that f possesses one zero in the interval (0,1). Here the function f shall satisfy (Hf1). Working with frequencies we add the conditions

(Hg1) $\qquad 0 \leq g(x) \leq 1, \quad x \in \overline{\mathbb{R}^+},$

(Hh1) $\qquad 0 \leq h(t) \leq 1, \quad t \in \overline{\mathbb{R}^+},$

and further we require some smoothness in order to be sure that the solution $u(x,t;g,h)$ of (P) is a classical one: $u \in C^{2,\alpha}(\bar{Q})$:

$$
\left.
\begin{array}{ll}
\text{(Hg2)} & g \in C^{2,\alpha}(\overline{\mathbb{R}^+}) \\
\text{(Hh2)} & h \in C^{1,\alpha/2}(\overline{\mathbb{R}^+})
\end{array}
\right\} \quad \text{for some } \alpha,\ 0 < \alpha < 1
$$

and the consistency conditions

$$
\text{(Hgh3)} \qquad
\begin{array}{l}
h(0) = g(0), \\
h'(0) = g''(0) + f(g(0)).
\end{array}
$$

In Section 2 we gathered some notations and gave the precise formulation of the existence and uniqueness of the solution of problem (P).

For the pushed case we shall prove globally the same results as in VELING [12] under quite general conditions on g and h. It means that the solution $u(x,t;g,h)$ converges to an asymptotic state composed of

(i) a solution $V_\theta(x)$ of the stationary problem, namely

$$
(1.8) \qquad
\begin{array}{l}
V_\theta'' + f(V_\theta) = 0, \quad x \in \mathbb{R}^+, \\
V_\theta(0) = \theta, \quad \lim_{x\to\infty} V_\theta(x) = 1, \quad 0 \le \theta \le 1,
\end{array}
$$

and

(ii) some translate of the travelling wave $U_{c(f)}$.

The conditions on f, g and h are as follows:

(Hf2) $\quad c(f) > 2\sqrt{f'(0)}$ (i.e. the pushed case),

(Hg4) $\quad \exists M > 0, \quad \exists \lambda > -\beta_3(c(f)) \cdot\ni\cdot\ g(x) \le Me^{-\lambda x}, \quad x \ge 0,$

(Hh4) $\quad \exists \theta, \quad 0 \le \theta \le 1, \quad \exists L > 0,\ \exists \gamma > 0 \cdot\ni\cdot\ |\theta - h(t)| \le Le^{-\gamma t}, \quad t \ge 0$

and a very simple threshold condition, namely

$$\text{(Hgh5)} \qquad \exists x_0 \cdot\ni\cdot\ g(x_0) > 0 \quad \text{or} \quad \exists t_0 \cdot\ni\cdot\ h(t_0) > 0,$$

so we are assured that g and h are not both identical zero. The techniques we use, do not seem strong enough to prove the same results for the pulled case. But, in view of the fact that in the corresponding Cauchy problem with a step function as initial condition, the convergence to a travelling wave is only in form, it is clear that this case is a more delicate one. For this problem BRAMSON [2] has given a beautiful asymptotic result based on the theory of Brownian motion.

Let us now formulate the result of the present paper.

THEOREM 1.1. *Let the conditions* (Hf1-2), (Hg1-5), (Hh1-5) *be satisfied. Then there exist constants* z_0, K, ω, $K > 0$, $\omega > 0$ *such that the solution* $u(x,t;g,h)$ *of problem* (P) *satisfies*

$$(1.9) \qquad |u(x,t;g,h) - U_{c(f)}(x - c(f)t - z_0) - V_\theta(x) + 1| < Ke^{-\omega t},$$

$$\text{uniformly } x \geq 0,\ t \geq 0,$$

where $c(f)$ *is the minimal velocity corresponding with the nonlinearity* f.

In contrast with the "heterozygote inferior" case (Veling [12]) there is now no need for a threshold condition except for the trivial one (Hgh5). Here we need further the knowledge of the rate of decay of the initial condition for $x \to \infty$. The comparable condition in [12] is $\limsup_{x\to\infty} g(x) < a$, where $u = a$ is the zero between 0 and 1 of the function f, so it is obvious that there the class of admissible initial conditions is broader.

The proof of this result runs mainly along the lines of the proof of FIFE & McLEOD [3] for their corresponding result. We have to make a modification for the fact that there now exists a half-line of possible velocities. The necessary modification is due to ROTHE [9]. Finally we apply the same techniques as used in VELING [12] for treating an initial-boundary value problem. This involves splitting up the domain $\bar{Q} = (\overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+})$ into two parts: $\bar{Q} = \overline{Q^1} \cup \overline{Q^2}$, where

$$(1.10)\qquad Q^1 = \{(x,t) \mid x > c_1 t,\ t > 0\},$$

$$(1.11)\qquad Q^2 = \{(x,t) \mid 0 < x < c_1 t,\ t > 0\},$$

for some $c_1 < c(f)$, and proving that for $t \to \infty$, $u \to U_{c(f)}$ in Q^1 and $u \to V_\theta$ in $\overline{Q^2}$, both exponentially. Together these results prove that u converges to an asymptotic state (1.9).

The first step in the proof is to consider the case $\theta = 1$. Then there is no need to split up the domain Q, but we consider for fixed $\delta > 0$.

$$(1.12)\qquad Q_\delta = \{(x,t) \mid x > \delta,\ t > \delta\}.$$

With the choice of suitable sub- and supersolutions it is possible to obtain a priori bounds on the solution and its derivatives. With this knowledge we can prove that the orbit $\{u(\cdot,t) \mid t \geq \delta\}$ is relatively compact in $C^2([\delta,\infty))$ and, with the aid of a Lyapunov-functional that the limit point in $C^2([\delta,\infty))$ is equal to a translate of $U_{c(f)}$ (section 3).

For the general case $0 \leq \theta < 1$ we consider $\overline{Q}$ as $\overline{Q} = \overline{Q^1} \cup \overline{Q^2}$ and we apply the first result to Q^1. Therefore it is necessary to know the behaviour of the solution u along the line $x = c_1 t$ in the (x,t)-plane. With a complicated lowersolution we prove that $1 - u(c_1 t,t) = O(e^{-\gamma t})$, $t \to \infty$ for some $\gamma > 0$ (section 4). Further we prove that $u \to V_\theta$ exponentially in $\overline{Q^2}$ in an analogous way as in section 3 (section 5). This same technique is also used for the case $\theta = 1$ and this completes the last part of the proof (section 6).

<u>EXAMPLE</u>. If we consider the canonical nonlinearity f_F (see (1.2)) and we transform as follows: $x' = \sqrt{1-\tau}\, x$, $t' = (1-\tau)t$, $\nu = -1 + (\sigma-1)/(1-\tau)$, then we find, dropping the accents again

$$(1.13)\qquad u_t = u_{xx} + \tilde{f}(u), \qquad \tilde{f}(u) = u(1-u)(1+\nu u).$$

The "heterzygote intermediate" case corresponds with values of ν such that $\nu > -1$. It is possible to calculate the minimal velocity $c(\tilde{f})$ as a function of the parameter ν, see HADELER & ROTHE [5]

$$(1.14)\qquad c(\tilde{f}) = \begin{cases} 2, & -1 < \nu \le 2, \\ \sqrt{\frac{\nu}{2}} + \sqrt{\frac{2}{\nu}}, & 2 \le \nu. \end{cases}$$

We note that $\tilde{f}'(0) = 1$, so from (1.14) it follows that for $\nu > 2$ condition (Hf2) has been satisfied for the choice $f = \tilde{f}$. It is even possible to give an explicit representation of $U_{c(\tilde{f})}(z)$ for $\nu \ge 2$, i.e. the so-called Huxley wave

$$(1.15)\qquad U_{c(\tilde{f})}(z) = \frac{1}{1+e^{\sqrt{\nu/2}\,z}}, \qquad z \in \mathbb{R}.$$

For $f = \tilde{f}$ we can also give a representation of the solution of (1.8), namely

$$(1.16)\qquad V_\theta(x) = 1 - \frac{(2\nu+2)}{\frac{2}{3}(1+2\nu) + \frac{1}{3}\sqrt{2\nu^2+2\nu-4}\,\sinh(\sqrt{\nu+1}\,x+B)}$$

$$(1.17)\qquad B = \operatorname{arsinh}\left(\frac{3(2\nu+2)-(4\nu+2)(1-\theta)}{(1-\theta)\sqrt{2\nu^2+2\nu-4}}\right)$$

and the asymptotic behaviour for V_θ

$$(1.18)\qquad 1 - V_\theta(x) = O(e^{-\sqrt{\nu+1}\,x}), \qquad x \to \infty.$$

For general f the exponent $\sqrt{\nu+1}$ should be changed into $\sqrt{-f'(1)}$.

2. NOTATIONS AND PRELIMINARIES

We introduce the following notations and definitions:

$I = (a,b) \subset \mathbb{R}$, $-\infty \le a < b \le \infty$,

$C^m(I) = \{u=u(x) \mid u$ m-times continuously differentiable, $x \in I\}$,

$\bar{I}$ compact: $C^m(\bar{I}) = \{u=u(x) \mid u$ m-times continuously differentiable, $u, u^{(1)},\dots,u^{(m)}$ bounded in I and $u,u^{(1)},\dots,u^{(m)}$ can be extended to continuous functions on $\bar{I}\}$,

$\bar{I}$ not compact: $C^m(\bar{I}) = \{u{=}u(x) \mid u$ m-times continuously differentiable and $u, u^{(1)}, \ldots, u^{(m)}$ bounded and uniformly continuous in $I\}$,

$$|u|_0^I = \sup_{x \in I} |u(x)|, \qquad |u|_m^I = \sum_{i=0}^{m} |u^{(i)}|_0^I ,$$

$$H(u;\alpha;I) = \sup_{x_1, x_2 \in I; x_1 \neq x_2} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha}, \qquad 0 < \alpha \leq 1,$$

$$C^{m,\alpha}(\bar{I}) = \{u{=}u(x) \mid u \in C^m(\bar{I}),\ H(u^{(m)};\alpha;I) < \infty\},$$

$$|u|_{m,\alpha}^I = |u|_m^I + \sum_{i=0}^{m} H(u^{(i)};\alpha;I).$$

Next we define some classes of functions depending on the spatial argument x and on the time t, $(x,t) \in D$, D open.

$$C^0(D) = \{u{=}u(x,t) \mid u \text{ continuous, } (x,t) \in D\},$$

$$C^1(D) = \{u{=}u(x,t) \mid u,\ u_x \text{ continuous, } (x,t) \in D\},$$

$$C^2(D) = \{u{=}u(x,t) \mid u,\ u_x,\ u_{xx},\ u_t \text{ continuous, } (x,t) \in D\},$$

and as for a scalar variable: $C^0(\bar{D})$, $C^1(\bar{D})$, $C^2(\bar{D})$,

$$d(P_1,P_2) = \{(x_1-x_2)^2 + |t_1-t_2|\}^{\frac{1}{2}}, \quad \text{with } P_i(x_i,t_i), \quad i = 1,2,$$

$$u(P) = u(x,t) \quad \text{for} \quad P = (x,t),$$

$$|u|_0^D = \sup_{P \in D} |u(P)|,$$

$$H(u;\alpha;D) = \sup_{P_1,P_2 \in D; P_1 \neq P_2} \frac{|u(P_1) - u(P_2)|}{d(P_1,P_2)^\alpha}, \qquad 0 < \alpha \leq 1,$$

$$C^{2,\alpha}(\bar{D}) = \{u{=}u(x,t) \mid u \in C^2(\bar{D}),\ H(u_{xx};\alpha;D) < \infty,\ H(u_t;\alpha;D) < \infty\},$$

$$|u|_\alpha^D = |u|_0^D + H(u;\alpha;D),$$

$$|u|^D_{2,\alpha} = |u|^D_\alpha + |u_x|^D_\alpha + |u_{xx}|^D_\alpha + |u_t|^D_\alpha .$$

Further we define

$$\text{(2.1)} \qquad \chi(y) = \frac{\int_{-1}^{y} e^{1-\frac{1}{1-s^2}} ds}{\int_{-1}^{1} e^{1-\frac{1}{1-s^2}} ds}, \qquad -1 \le y \le 1.$$

As is noted in section 1, we study travelling wave solutions, i.e. solutions depending only on the variable $z = x - ct$. In general we shall use the notation

$$\text{(2.2)} \qquad u(x,t) = u(z+ct,t) \equiv v(z,t) = v(x-ct,t),$$

where $v(z,t)$ satisfies

$$\text{(2.3)} \qquad v_t = v_{zz} + cv_z + f(v)$$

whenever $u(x,t)$ satisfies

$$\text{(2.4)} \qquad u_t = u_{xx} + f(u).$$

In the sequel we need the following theorem.

<u>THEOREM 2.1</u>. *A Priori Estimate Theorem (See* [3]*). Let* $Q = (a,b) \times (t_0,t_1)$, $t_0 \ge 0$, $-a$, b, t_1 *possibly infinite. Let* $Q_\delta = (a+\delta,b-\delta) \times (t_0+\delta,t_1)$, $0 < \delta < \min((b-a)/2, t_1-t_0)$. *Let* $w \in C^2(Q)$ *and let* w *satisfy* $w_t = w_{xx} + cw_x + f(w)$, $(x,t) \in Q$, *with* $|w|^Q_0 \le K$ *and* $f \in C^{0,1}([-K,K])$. *Then the following estimates hold for some* α, $0 < \alpha < 1$, *where the constant* C *depends only on* δ *and* α

$$\text{(2.5)} \qquad |w|^{Q_\delta}_0 + |w_x|^{Q_\delta}_0 \le C(|w^Q_0| + |f\circ w|^Q_0),$$

$$\text{(2.6)} \qquad |w|^{Q_\delta}_0 + |w_x|^{Q_\delta}_0 + |w_{xx}|^{Q_\delta}_0 + |w_t|^{Q_\delta}_0$$

$$\le C\{H(f\circ w;1;Q)(|f\circ w|^Q_0 + |w|^Q_0) + |w|^Q_0\},$$

$$(2.7)\qquad H(w_{xx};\alpha;Q_\delta),H(w_t;\alpha;Q_\delta)$$

$$\le C\{H(f\circ w;1;Q)(|f\circ w|_0^Q + |w|_0^Q) + |w|_0^Q\}.$$

In this formulation it is possible to apply this theorem both for the (x,t)-arguments $(c=0)$ and for the (z,t)-arguments $(c\neq 0)$.

THEOREM 2.2. *Existence and Uniqueness. Let the conditions* (Hf1), (Hg1-3), (Hh1-3) *be satisfied, then problem* (P) *has a unique solution* $u \in C^{2,\alpha}(\bar{Q})$.

PROOF. We use a theorem in OLEINIK & KRUHZKOV ([8]; Theorem 14). They treat the corresponding Cauchy problem. Our conditions (Hg2-3), (Hh2-3) give the required smoothness, which is necessary for handling this initial-boundary value problem.

3. THE CASE $\theta = 1$

In this section we shall prove that for $\theta = 1$ the solution u converges uniformly and exponentially for $x \ge \delta > 0$ to $U_{c(f)}$ as $t \to \infty$. The main part of the proof will consist of the following lemma.

LEMMA 3.1. *Let the conditions* (Hf1-2), (Hg1-4), (Hh1-3) *be satisfied. Let condition* (Hh4) *be satisfied with* $\theta = 1$, *then the solution* $u(x,t;g,h)$ *of problem* (P) *can be bounded a priori as follows*

$$(3.1)\qquad \underline{u}(x,t) \le u(x,t;g,h), \quad x \ge 0,\ t \ge T_0,$$

$$(3.2)\qquad u(x,t;g,h) \le \bar{u}(x,t), \quad x \ge 0,\ t \ge 0,$$

where

$$(3.3)\qquad \underline{u}(x,t) = \max(0,U_{c(f)}(x-c(f)(t-T_0)+s(t)) - q(t)r(z)),$$

$$(3.4)\qquad \bar{u}(x,t) = \min(1,U_{c(f)}(x-c(f)t - \bar{s}(t)) + \bar{q}(t)\bar{r}(z)),$$

with

$$(3.5.ab)\qquad s(t) = s(T_0) + A(1-e^{-\beta(t-T_0)}),\quad \bar{s}(t) = \bar{s}(0) + \bar{A}(1-e^{-\beta t}),$$

$$(3.6ab)\qquad q(t) = q(T_0)e^{-\beta(t-T_0)},\quad \bar{q}(t) = \bar{q}(0)e^{-\beta t},$$

$$(3.7ab)\qquad r(z) = \begin{cases} 1 & , z \le -c(f)T_0, \\ e^{-\lambda_1(z+c(f)T_0)} & , z \ge -c(f)T_0, \end{cases}\qquad r(z) = \begin{cases} 1 & , z \le 0, \\ e^{-\lambda_1 z} & , z \ge 0. \end{cases}$$

REMARK. The constants β, T_0, $s(T_0)$, $\bar{s}(0)$, A, $\bar{A}$, $q(T_0)$, $\bar{q}(0)$ and λ_1 will be specified in the proof of the lemma. Except maybe for $s(T_0)$, $\bar{s}(0)$ they are all positive. Observe that the subsolution consists of a wave which travels with a lower speed than the speed connected with that wave-form, but the difference in speeds becomes exponentially small for increasing t. From this wave we subtract a positive function, the product of a uniformly in t decreasing function $q(t)$ and a function $r(z)$, which attenuates $q(t)$ for large z. The introduction of this factor $r(z)$ is a necessary modification of the sub- and supersolutions introduced by FIFE & McLEOD [3] for the case $f'(0) < 0$. Here we deal with the case $f'(0) < 0$ and so we need an adjustment.

PROOF. We shall use the maximum principle. In order that $\underline{u}$ is a subsolution we have to show that

(i) $\qquad L[\underline{u}] \equiv \underline{u}_{xx} + f(\underline{u}) - \underline{u}_t \ge 0, \qquad (x,t) \in Q,$

(ii) $\qquad \underline{u}(x,0) \le g(x) \qquad , \qquad x \in \overline{\mathbb{R}^+},$

(iii) $\qquad \underline{u}(0,t) \le h(t) \qquad , \qquad t \in \overline{\mathbb{R}^+}.$

In the sequel we shall use the shorthand notation $x = x(f)$, $\beta_i = \beta_i(c(f))$, $i = 1,2,3$, $U = U_{c(f)}(x-c(f)(t-T_0)+s(t))$, $s = s(t)$, $q = q(t)$, $r = r(z)$. It will turn out that it is not obvious that we can fulfil all the requirements necessary for proving (i), (ii), (iii). Especially for small x and t the points (ii), (iii) give conditions on the parameters in (3.5a), (3.6a), (3.7a) which seem to be contradictory. We shall circumvent this by the following series of observations. From condition (Hh4) we learn that for $t > T_1$, where

(3.8) $$T_1 = \frac{1}{\gamma} \ell n L,$$

the function h can be bounded below by

(3.9) $$h(t) \geq 1 - Le^{-\gamma t} > 1 - Le^{-\gamma T_1} = 0.$$

Let us consider the problem (P) for $(x,t) \in Q$ and $t > T_1$ with instead of the functions $h(t)$ and $g(x)$ the corresponding functions

(3.10) $$\tilde{h}(t) = 1 - Le^{-\gamma t}, \qquad t \geq T_1,$$

(3.11) $$\tilde{g}(x) = 0, \qquad x \geq 0.$$

The solution $\tilde{u}$ of this problem $(\tilde{P})$ satisfies trivially $\tilde{u} \leq u$ for $t > T_1$. Now we apply a theorem of ARONSON & WEINBERGER ([1], Theorem 5.1) to learn that u converges uniformly on bounded intervals to the function $u \equiv 1$. In view of this fact and the knowledge that $u \geq \tilde{u}$ we can make the following statement

(3.12) $$\forall p,\ 0 < p < 1,\ \forall X > 0,\ \exists T_2 = T_2(p,X) \ni u(x,T_2) \geq \tilde{u}(x,T_2) > p \quad \text{for } 0 \leq x \leq X.$$

When we need the fact that u is not any longer near zero for large time, we shall use (3.12). Further we need the following estimates with respect to the function f, see also FIFE & McLEOD [3] and VELING ([12], formula (4.4)) for analogous results. For convenience we extend the domain of f as follows

(3.13) $$\bar{f}(u) = \begin{cases} f'(0)u, & u \leq 0, \\ f(u) & 0 \leq u \leq 1, \\ f'(1)(u-1), & 1 \leq u, \end{cases}$$

then there exists a constant $K > 0$ such that

(3.14) $$\begin{aligned} &\bar{f}(u-q) - f(u) \geq -Kq, \quad 0 \leq q \leq 1,\ 0 \leq u \leq 1, \\ &\bar{f}(u+q) - f(u) \leq Kq, \quad 0 \leq q \leq 1,\ 0 \leq u \leq 1. \end{aligned}$$

For any choice of μ_1, $f'(0) < \mu_1 \le K_1$, there exists a $\delta_1 > 0$, such that

(3.15)
$$\bar{f}(u-q) - f(u) \ge -\mu_1 q, \quad 0 \le q \le 1,\ 0 \le u \le \delta_1,$$
$$\bar{f}(u+q) - f(u) \le \mu_1 q, \quad 0 \le q \le 1,\ 0 \le u \le \delta_1,$$

and further for any choice of q_2, there exist positive numbers μ_2, δ_2 such that

(3.16)
$$f(u-q) - f(u) \ge \mu_2 q, \quad 0 \le q \le q_2,\ 1-\delta_2 \le u \le 1,$$
$$\bar{f}(u+q) - f(u) \le -\mu_2 q, \quad 0 \le q \le 1,\ 1-\delta_2 \le u \le 1,$$

with $q_2 + \delta_2 < 1$ and $\mu_2 < -f'(1)$. We remark that the expression

(3.17)
$$\ell = \sup_{\delta_1 \le U \le 1-\delta_2} \frac{d}{dz} U_{c(f)}(z)$$

is negative and bounded away from zero. For small values of δ_1, δ_2, ℓ behaves as

(3.18)
$$\ell = \max(\beta_2 \delta_1, -\beta_1(1-\delta_2))(1+o(1)), \quad \delta_1 \downarrow 0,\ \delta_2 \downarrow 0,$$

see (1.5) for β_1, (1.6) for β_2, both evaluated for $c = c(f)$.

We calculate $L[\underline{u}]$:

(3.19)
$$z < -c(f)T_0:\ L[u] = f(U-q) - f(U) - \dot{s}U_z + \dot{q},$$

(3.20)
$$-c(f)T_0 < z < z_1:\ L[\underline{u}] = f(U-qr) - f(U) - \dot{s}U_z - qr_{zz} - cqr_z + \dot{q}r,$$

(3.21)
$$z_1 < z:\ L[\underline{u}] = 0,$$

where $\cdot$ denotes $\frac{d}{dt}$ and $z_1 = x_1 - c(f)t$ is defined by $u(x_1,t) = 0$. We proceed the calculations by inserting the expressions (3.5a), (3.6a), (3.7a) into (3.19) and (3.20); we distinguish thereby the cases $0 \le U \le \delta_1$, $\delta_1 \le U \le 1-\delta_2$, $1-\delta_2 \le U \le 1$ in order to prove that $L[\underline{u}] \ge 0$.

<u>CASE 1</u>. $z \le -c(f)T_0$, $0 \le U \le \delta_1$. We choose the parameters in such a way that this case will not occur, so for $z < -c(f)T_0$ and $t \ge T_0$

$$U = U_{c(f)}(x-c(f)(t-T_0)+s(t)) > U_{c(f)}(s(T_0)+A).$$

We require

(3.22) $$U_{c(f)}(s(T_0)+A) > \delta_1.$$

<u>CASE 2</u>. $z \le -c(f)T_0$, $\delta_1 \le U \le 1-\delta_2$. We estimate $L[\underline{u}]$ as follows:

$$L[\underline{u}] \ge -Kq + \beta A e^{-\beta(t-T_0)}(-\ell) - \beta q = q(-K + \beta A(-\ell)/q(T_0) - \beta) \ge 0$$

by choosing

(3.23) $$A = \frac{K+\beta}{\beta(-\ell)}\, q(T_0).$$

<u>CASE 3</u>. $z \le -c(f)T_0$, $1-\delta_2 \le U \le 1$. We estimate $L[\underline{u}]$ as follows:

$$L[\underline{u}] \ge \mu_2 q - \beta q \ge 0$$

by choosing

(3.24) $$\beta \le \mu_2.$$

<u>CASE 4</u>. $-c(f)T_0 \le z \le z_1$, $0 \le U \le \delta_1$. We estimate $L[\underline{u}]$ as follows:

$$\begin{aligned} L[\underline{u}] &\ge f(U-qr) - f(U) - qr_{zz} - cqr_z + \dot{q}r \\ &\ge -\mu_1 qr - \lambda_1^2 qr + c\lambda_1 qr - \beta qr \\ &= qr[-\lambda_1^2 + c\lambda_1 - f'(0) + f'(0) - \beta - \mu_1]. \end{aligned}$$

We make the following choice for λ_1:

(3.25) $\quad -\beta_3 < \lambda_1 < \min(\lambda,-\beta_2)$.

See (1.6) for β_2, (1.7) for β_3. By this choice (3.25) we have $-\lambda_1^2 + c\lambda_1 - f'(0) = p > 0$, so making the choice $\mu_1 = f'(0) + \frac{1}{2}p$ in (3.15) and taking

(3.26) $\quad \beta < \frac{1}{2}p$,

we find $L[\underline{u}] > 0$.

<u>CASE 5</u>. $-c(f)T_0 \le z \le z_1$, $\delta_1 \le U \le 1-\delta_2$. We estimate $L[\underline{u}]$ as follows:

$$L[\underline{u}] \ge -Kqr + \beta A e^{-\beta(t-T_0)}(-\ell) - \lambda_1^2 qr + c\lambda_1 qr - \beta qr$$

$$= qr(-K + \beta A(-\ell)/q(T_0) - \lambda_1^2 + c\lambda_1 - f'(0) + f'(0) - \beta)$$

$$\ge qrf'(0) > 0$$

by the choice (3.23) for A and the fact that $-\lambda_1^2 + c\lambda_1 - f'(0)$ is positive.

<u>CASE 6</u>. $-c(f)T_0 \le z \le z_1$, $1-\delta_2 \le U \le 1$. We estimate $L[\underline{u}]$ as follows:

$$L[\underline{u}] \ge f(U-qr) - f(U) - qr_{zz} - cqr_z + \dot{q}r$$

$$\ge \mu_2 qr - \lambda_1^2 qr + c\lambda_1 qr - \beta qr$$

$$= qr(-\lambda_1^2 + c\lambda_1 - f'(0) + f'(0) - \beta + \mu_2)$$

$$\ge qrf'(0) > 0$$

by the choice (3.24).

At this moment we have not yet specified δ_2, $q(T_0)$ other than as positive constants. Let us make the choice

(3.27) $\quad \delta_2 = q(T_0) = \delta_1$, where δ_1 follows from the choice for μ_1 under case 4 (see (3.15)), then also μ_2 and ℓ are determined. We choose

$$(3.28) \qquad \beta < \min(\gamma, \tfrac{1}{2}(-\lambda_1^2 + c\lambda_1 - f'(0)), \mu_2, \beta_1 c),$$

then also the number A is specified. We choose $s(T_0)$ so large that we can fulfil (3.22). Because the other conditions (ii) and (iii) for the proof that $\underline{u}$ is a subsolution requires a shift of U to the left, while (3.22) requires a shift to the right, it is not clear whether it is possible to meet them both. To avoid this we shall study problem (P) from $t \geq T_0$ onwards, where we use (3.12). We shall specify the size of T_0. From (1.6) we learn that there exists some number x_1 depending on $s(T_0)$ and $q(T_0)$ such that $\underline{u}(x,T_0) = 0$ for $x \geq x_1$ in view of the fact that $-\beta_2 > \lambda_1$. Remark that $\underline{u}(x,T_0)$ does not depend on T_0 by the definition of $\underline{u}$ ((3.3), (3.5a), (3.6a), (3.7a)). By (3.12) it is possible to choose a number T_2 such that for $p = U(s(T_0))$ and $X = x_1$

$$0 < \underline{u}(x,T_0) = U(x+s(T_0)) - q(T_0)e^{-\lambda_1 x}$$

$$< U(s(T_0)) = p < \tilde{u}(x,T_2), \qquad 0 \leq x \leq x_1 = X.$$

By the choice $T_0 \geq T_2$ we can meet condition (ii) for $t = T_0$. Note that $\tilde{u}(x,T_2) \leq \tilde{u}(x,T_0) \leq u(x,T_0)$. Further we choose a number $T_4 \geq 0$ such that

$$(3.29) \qquad T_4 = \begin{cases} \frac{1}{\beta} \ell n \frac{L}{q(T_0)}, & L \geq q(T_0), \\ 0 & L \leq q(T_0), \end{cases}$$

see for L (Hh4). We have for $t > T_4$ and $T_0 \geq T_4$

$$\underline{u}(0,t) = U(-c(t-T_0) + s(t)) - q(T_0)e^{-\beta(t-T_0)}$$

$$\leq U - 1 + 1 - q(T_0)e^{-\beta(t-T_0)} < 1 - q(T_0)e^{-\beta(t-T_4)}$$

$$\leq 1 - Le^{-\gamma t} \leq h(t),$$

by (3.28) and (3.29). So by the choice

$$(3.30) \qquad T_0 = \max(T_3,T_4)$$

we can also meet condition (iii) and finally we have proven that $\underline{u}$ is a subsolution.

The proof that $\bar{u}$ is a supersolution runs quite analogously, only the corresponding conditions (ii)' and (iii)' need some attention. We have to prove

$$\text{(ii)'} \qquad \bar{u}(x,0) \geq g(x), \quad x \in \overline{\mathbb{R}^+},$$

$$\text{(iii)'} \qquad \bar{u}(0,t) \geq h(t), \quad t \in \overline{\mathbb{R}^+}.$$

Because $\lambda_1 < -\beta_2$ we know that $\bar{u}(x,0) = O(e^{-\lambda_1 x})$, $x \to \infty$. By condition (Hg4) we derive that for some x_2, $\bar{u}(x,0) > g(x)$ for $x \geq x_2$, because also $\lambda_1 < \lambda$ (3.25). For the complement of the x-axis we define

$$(3.31) \qquad N = \max_{0 \leq x \leq x_2} g(x) - \bar{q}(0)e^{-\lambda_1 x}.$$

Choose now $\bar{s}(0)$ so large that

$$(3.32) \qquad U(x_2-\bar{s}(0)) \geq N,$$

then we have for $0 \leq x \leq x_2$

$$U(x-\bar{s}(0)) \geq U(x_2-\bar{s}(0)) \geq N \geq g(x) - \bar{q}(0)e^{-\lambda_1 x},$$

thus

$$\bar{u}(x,0) = U(x-\bar{s}(0)) + \bar{q}(0)e^{-\lambda_1 x} \geq g(x), \quad 0 \leq x \leq x_2.$$

We study $\bar{u}(0,t)$. For $t \geq 0$ we have by (1.5) for some constant $\tilde{C} > 0$

$$\bar{u}(0,t) - h(t) = \min(1, U(-ct-\bar{s}(t)) + \bar{q}(t)) - h(t)$$

$$\geq U(-ct-\bar{s}(0)) - 1 + \bar{q}(t) + 1 - h(t)$$

$$\geq -\tilde{C}e^{-\beta_1 ct - \beta_1 \bar{s}(0)} + \bar{q}(0)e^{-\beta t} \geq 0$$

by enlarging $\bar{s}(0)$ once more and by the choice $\beta < \beta_1 c$, which we had required already in (3.28). This implies condition (iii)' and finally we have proven (3.1) and (3.2). □

For convenience we introduce the following functions

$$(3.33) \qquad E_1(x,t) = e^{\beta_2(x-c(f)t)} + e^{-\beta t} e^{-\lambda_1(x-c(f)t)},$$

$$(3.34) \qquad E_2(x,t) = e^{\beta_1(x-c(f)t)} + e^{-\beta t}.$$

LEMMA 3.2. *Let the same conditions be satisfied as in Lemma* 3.1. *Let* δ *be an arbitrary positive number. Then the following estimates hold for the solution* $u(x,t)$ *of problem* (P) *for some positive constants* C_1, C_2:

$$(3.35) \qquad u(x,t) < C_1 E_1(x,t), \quad x \geq c(f)t,\ t \geq T_0,$$

$$(3.36) \qquad 1 - u(x,t) < C_1 E_2(x,t), \quad 0 \leq x \leq c(f)t,\ t \geq 0,$$

$$(3.37) \qquad |u_x(x,t)|,\ |u_{xx}(x,t)|,\ |u_t(x,t)|,$$

$$H(u_{xx};\alpha;Q_1),\ H(u_t;\alpha;Q_1) < C_2 E_1(x,t),$$

$$Q_1 = \{(x,t) \mid x \geq c(f)t,\ t \geq T_0 + \delta\},$$

$$(3.8) \qquad |u_x(x,t)|,\ |u_{xx}(x,t)|,\ |u_t(x,t)|,$$

$$H(u_{xx};\alpha;Q_2),\ H(u_t;\alpha;Q_2) < C_2 E_2(x,t),$$

$$Q_2 = \{(x,t) \mid \delta \leq x \leq c(f)t,\ t \geq \delta\},$$

where the value for α *follows from conditions* (Hg2), (Hh2).

PROOF. The asymptotic behaviour of U (1.5), (1.6) together with the bounds from Lemma 3.1 (3.1), (3.2) gives the estimates (3.35) and (3.36). An application of the A Priori Estimate Theorem (Theorem 2.1) gives (3.37) and (3.38). For the technique to establish a pointwise bound for the derivatives with the aid of this theorem see VELING [12]. □

It will be convenient to extend the domain of the function u(x,t) from $\overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+}$ to $\mathbb{R} \times \overline{\mathbb{R}^+}$. This will be done in the following way

$$(3.39) \qquad \tilde{u}(x,t) = \begin{cases} 1 \quad , & x \le \delta,\ t \ge 0, \\ \psi(x,t), & \delta \le x \le 2\delta,\ t \ge 0, \\ u(x,t), & 2\delta \le x,\ t \ge 0, \end{cases}$$

where $\psi(x,t)$ represents a smooth connection between the function u and 1 (see for χ (2.1)):

$$(3.40) \qquad \psi(x,t) = 1 + \chi(\tfrac{2x-3\delta}{\delta})(u(x,t) - 1).$$

Now the same estimates with maybe a larger constant C_3 instead of C_1, C_2 in formulas (3.35), (3.36), (3.37) and (3.38) are valid for the function $\tilde{u}$ in the larger domain, because the constant part of $\tilde{u}$ satisfies trivially these estimates and the function χ and its derivatives are bounded. We have found suitable estimates for proving the convergence of u(x,t) to $U_{c(f)}(x-c(f)t-z_0)$, i.e. some translate of a travelling wave. We formulate the result of this section as

THEOREM 3.1. *Let the conditions* (Hf1-2), (Hg1-4), (Hh1-3) *be satisfied. Let condition* (Hh4) *be satisfied with* $\theta = 1$. *Let* δ *be an arbitrary positive number, then there exist constants* z_0, K, ω, K > 0, ω > 0 *such that the solution* u(x,t;g,h) *of problem* (P) *satisfies*

$$(3.41) \qquad |u(x,t;g,h) - U_{c(f)}(x-c(f)t - z_0)| < Ke^{-\omega t},$$

uniformly $x \ge \delta > 0$, $t \ge 0$.

PROOF. We refer to FIFE & McLEOD ([3], Theorem 3.1). Once we have established the bounds connected with the travelling wave $U_{c(f)}$ (Lemma 3.2) the proof of this theorem is almost identical to that in Fife & McLeod. The fact that $f'(0) > 0$ instead of $f'(0) < 0$ needs some attention. At the proof of the exponential rate of convergence we deal with a linearized differential equation. Except for an eigenvalue zero which turns up by the translation invariancy of $U_{c(f)}$ we have to know that the rest of the spectrum consists of positive values and is bounded away from zero. The lower bound of the continuous spectrum reads $\min(\frac{1}{4} c^2(f)-f'(0), \frac{1}{4} c^2(f)-f'(1)) = \frac{1}{4} c^2(f) - f'(0)$, which is positive because we treat the pushed case. The rest of the discrete part of the spectrum is positive and lies between zero and $\frac{1}{4} c^2(f) - f'(0)$, so indeed except for an eigenvalue zero the spectrum is part of the positive real half-line and bounded away from zero. The fact that we have proven (3.35) only for $t \geq T_0$ has been incorporated in the formulation of the result (3.41) by enlarging the constant K. □

4. THE GENERAL CASE $0 \leq \theta < 1$; BEHAVIOUR OF u IN Q^1

We wish to apply the result of section 3 in the domain $Q^1 = \{(x,t) \mid x > c_1 t, t > 0, c_1 < c(f)\}$, see figure 1. It means that now the function u itself will play the role of the boundary function h. For an application of Theorem 3.1 we have to prove that $u(c_1 t,t)$ has the same properties as the function h in condition (Hh4), with $\theta = 1$. With the aid of a complicated subsolution we can prove that indeed u satisfies this required condition.

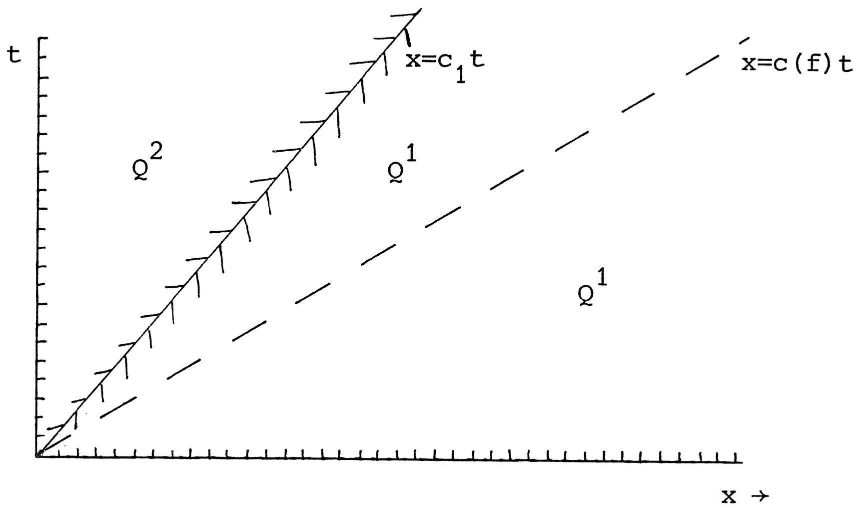

Fig. 1

We formulate this result as

<u>LEMMA 4.1</u>. *Let the conditions* (Hf1-2), (Hg1-5), (Hh1-5) *be satisfied. Then for the solution* u(x,t) *of problem* (P) *the following estimate holds for some positive constants* C^*, γ^* *and* $c_1 < c(f)$

$$(4.1) \qquad 1 - u(c_1 t,t) < C^* e^{-\gamma^* t}, \qquad t \geq 0.$$

<u>PROOF</u>. We construct a subsolution $\underline{m}(x,t)$. For the proof we refer to VELING ([12], Appendix), where we have dealt with the same function except for a small modification. This modification is the introduction of the function r(z) (3.7a), discussed in section 3. □

The subsolution is a composition of four functions $\underline{m}_i$, i = 1,2,3,4:

$$(4.2) \qquad \underline{m}_1(x,t) = 0,$$

$$(4.3) \qquad \underline{m}_2(x,t) = V_0(x-r(t)) - p_0 e^{\beta x-\alpha(x-T_0)},$$

$$(4.4) \qquad \underline{m}_3(x,t) = U_{c(f)}(x-c(f)(t-T_0)+s(t)) - q_0 e^{-\gamma(t-T_0)},$$

$$(4.5) \qquad \underline{m}_4(x,t) = U_{c(f)}(x-c(f)(t-T_0)+s(t)) - q_0 e^{-\gamma(t-T_0)} e^{-\lambda_1(x-c(f)(t-T_0)+s(T_0))},$$

where r(t) and s(t) are defined by

$$(4.6) \qquad r(t) = R(1 - e^{-\gamma(t-T_0)}),$$

$$(4.7) \qquad s(t) = s(T_0) + S(1 - e^{-\gamma(t-T_0)}).$$

See (1.8) for the definition of $V_0(x)$. We define the subsolution $\underline{m}$ as follows

$$(4.8)\qquad \underline{m}(x,t) = \begin{cases} \max(\underline{m}_1,\underline{m}_2) = \underline{m}_1, & 0 \le x \le x_1(t), \\ \max(\underline{m}_2,\underline{m}_3) = \underline{m}_2, & x_1(t) \le x \le x_2(t), \ \underline{m}_{2x}(x_2(t),t) < 0, \\ \max(\underline{m}_3,\underline{m}_4) = \underline{m}_3, & x_2(t) \le x \le x_3(t) = c(f)(t-T_0)-s(T_0), \\ \max(\underline{m}_4,\underline{m}_1) = \underline{m}_4, & x_3(t) \le x \le x_4(t), \\ \underline{m}_1 \qquad\qquad , & x_4(t) \le x. \end{cases}$$

The function $\underline{m}(x,t)$ has been pictured in figure 2. For the specification of the positive parameters T_0, p_0, α, β, γ, q_0, R, S and the parameter $s(T_0)$ we refer to VELING [12]; λ_1 has the same value as in (3.25). In the same manner as in section 3 we start with the construction of the subsolution (4.8) at a time T_0, where we invoke an analogous result as (3.12):

$$(4.9)\qquad \forall\varepsilon > 0 \quad \forall X > 0 \quad \exists T_0 = T_0(\varepsilon,X) > 0 \quad \cdot\ni\cdot$$

$$0 < V_0(x) - \tilde{u}(x,T_0) < \varepsilon \quad \text{for } 0 < x \le X.$$

Here $\tilde{u}$ satisfies problem (P) with $h \equiv 0$ and $g \not\equiv 0$ for $t \ge T_\varepsilon$. T_ε, $Y = \operatorname{supp} g$ and $\sup_{x\in Y} g(x)$ are all arbitrarily small. $\underline{m}(x,t)$ is a subsolution for $\tilde{u}(x,t)$ which is in his turn a subsolution for $u(x,t)$.

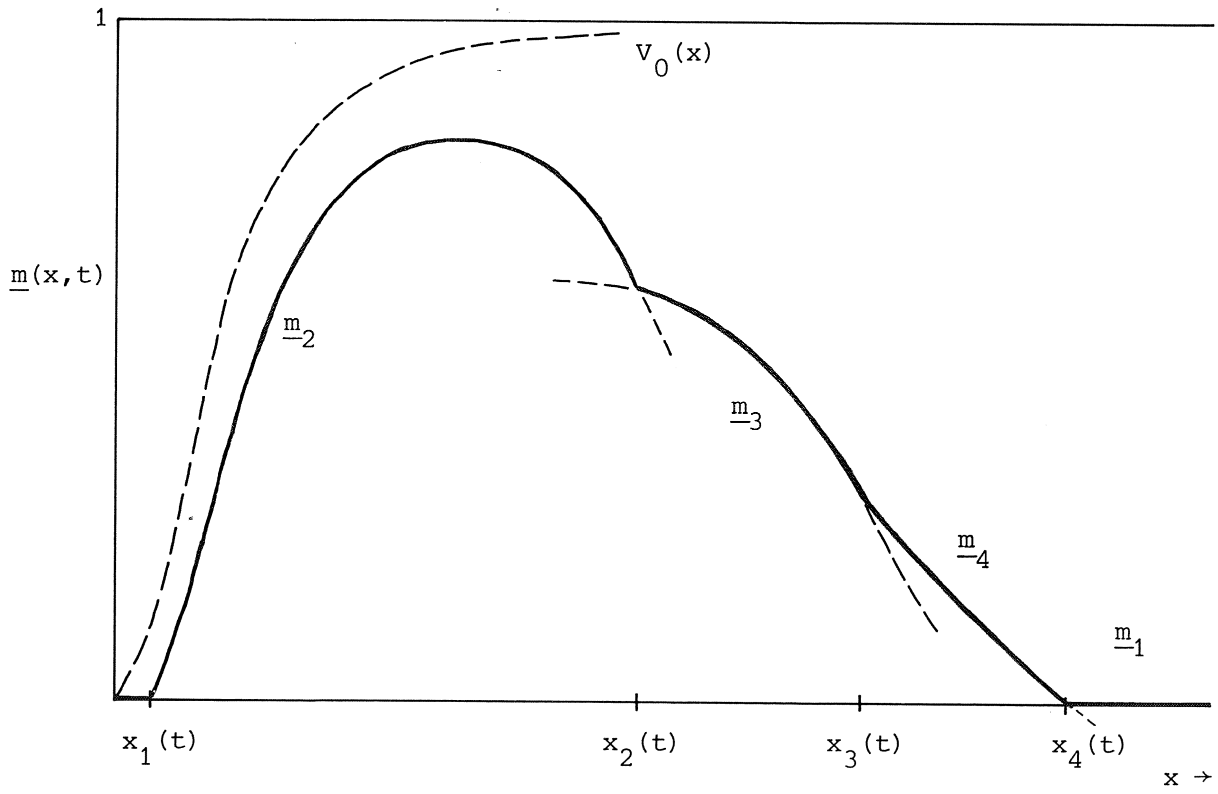

Fig. 2

A description of this subsolution runs as follows. From the stationary solution $V_0(x)$ we subtract a travelling wave with wavespeed α/β. This travelling wave can be represented by the variable $z' = x - (\alpha/\beta)t$ as

$$p(z') = p_0 e^{\beta z'}. \tag{4.10}$$

For some positive number p_2 such that $p(z') < p_2$ it can be proved that $\underline{m}_2$ is a subsolution. In view of the fact that (4.9) gives only for bounded intervals a statement about the behaviour of $u(x,t)$ our subsolution has to be bent back to the zero-level. This can be done with the aid of the same subsolution as we have used in section 3. In this manner we have found a subsolution with a support expanding in time. In fact the subsolution $\underline{m}$ imitates the behaviour of the function u for large time in case that $\theta = 0$. Now we apply Theorem 3.1 in the domain Q^1.

THEOREM 4.1. *Let the conditions* (Hf1-2), (Hg1-5), (Hh1-5) *be satisfied. Then there exist for arbitrary* $\delta > 0$ *constants* z_0, K^*, ω^*, $K^* > 0$, $\omega^* > 0$, *such that the solution* $u(x,t;g,h)$ *of problem* (P) *satisfies for* $c_1 < c(f)$

$$|u(x,t;g,h) - U_{c(f)}(x - c(f)t - z_0)| < K^* e^{-\omega^* t}, \tag{4.11}$$

$$\text{uniformly } x \geq c_1 t + \delta,\ t \geq 0.$$

PROOF. We apply Theorem 3.1 and use Lemma 4.1. We remark that the fact that the lower bound of x of the domain Q^1 depends on t, does not affect the proof, because we only need that U along this boundary tends exponentially to 1 and this is the case since $c_1 < c(f)$. □

REMARK. In condition (Hh4) it is allowed that $\theta = 1$, but then Theorem 4.1 gives less information than Theorem 3.1.

5. THE GENERAL CASE $0 \leq \theta < 1$; BEHAVIOUR OF u IN Q^2

In this section we shall prove that u converges exponentially to the function V_θ in $\overline{Q^2} = \{(x,t) \mid 0 \leq x \leq c_1 t,\ t \geq 0,\ c_1 < c(f)\}$, where θ is determined by the limit value of h.

THEOREM 5.1. *Let the conditions* (Hf1-2), (Hg1-5), (Hh1-3), (Hh5) *be satisfied. Let condition* (Hh4) *be satisfied with* $0 \le \theta < 1$, *then there exist positive constants* $\tilde{K}$, $\tilde{\omega}$ *such that the solution* $u(x,t;g,h)$ *of problem* (P) *satisfies*

$$(5.1)\qquad |u(x,t;g,h) - V_\theta(x)| < \tilde{K}e^{-\tilde{\omega}t}, \quad \text{uniformly } 0 \le x \le c_1 t + \delta,\ t \ge 0.$$

PROOF. For the proof we refer to VELING ([12], section 5). Here we give a short outline. A priori we can bound the solution below by the subsolution $\underline{m}(x,t)$, $(x,t) \in Q^2$. We define a new function $\tilde{u}(x,t)$ which is identical to $u(x,t)$ in $Q^2 \cup \{(x,t) \mid c_1 t \le x \le c_1 t + \delta,\ t \ge 0\}$ and which is identical to $V_\theta(x)$ in $\{(x,t) \mid c_1 t + 2\delta \le x,\ t \ge 0\}$. In the remaining strip we connect both functions with the aid of the C^∞-function χ (2.1). In this way the expression $1 - \tilde{u}(x,t)$ can be bounded for some positive constant C_1 as

$$(5.2)\qquad 1 - \tilde{u}(x,t) < C_1 e^{-\tau x}, \quad x \ge 0,\ t \ge T_0,$$

where τ is determined by the subsolution $\underline{m}(x,t)$. By invoking the A Priori Estimate Theorem (Theorem 2.1) and the fact that $u \in C^{2,\alpha}(\bar{Q})$ (Theorem 2.2) we can bound all the functions appearing in (3.38) (read $\tilde{u}$ for u) up till the boundary $x = 0$ for $t \ge T_0$ by the same exponential term as was used in (5.2), possibly with a large constant C_2. With a Lyapunov functional we prove that $\tilde{u}(x,t)$ converges to $V_\theta(x)$ and also that the rate of convergence is exponential by linearizing around $u = V_\theta(x)$. This will be done in the following way: define

$$(5.3)\qquad k(x,t) = \tilde{u}(x,t) - V_\theta(x - \alpha(t)),$$

where $\alpha(t)$ satisfies

$$(5.4)\qquad k(0,t) = h(t) - V_\theta(-\alpha(t)) = 0.$$

Then we study the spectrum of the differential operator

$$Mk = -k'' - f'(V_\theta(x))k, \quad k(0) = 0,\ k \in \mathcal{D}(M)$$

where M is self-adjoint and $\mathcal{D}(M) \subset L^2(\mathbb{R}^+)$ is an extension of $C_0^\infty(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. The spectrum $\sigma(M)$ consists of a continuum $[\bar{\lambda},\infty)$, $\bar{\lambda} = \lim_{x\to\infty} - f'(V_\theta(x)) = -f'(1) > 0$ and possibly a discrete part in $(-\lambda(\theta),\bar{\lambda})$, $\lambda(\theta) = \sup_{x\in\overline{\mathbb{R}^+}} f'(V\ (x))$. A priori for small values of θ there holds $\lambda(\theta) > 0$, so this estimate for the range of the discrete part of the spectrum includes $\lambda = 0$. But by the following reasoning we can prove that the lowest eigenvalue λ_0 exceeds zero and this gives enough information for establishing the exponential rate of convergence. Let the eigenfunction belonging to the eigenvalue λ_0 be represented by $\ell(x)$, if it exists, then we have

(5.5) $$-\ell'' - f'(V_\theta(x))\ell = \lambda_0\ell, \quad \ell(0) = 0,\ \ell(x) > 0,\ x \in \mathbb{R}^+.$$

We differentiate the equation for $V_\theta''(V_\theta\ \ + f(V_\theta) = 0)$:

(5.6) $$V_\theta''' + f'(V_\theta)V_\theta' = 0.$$

Multiply (5.5) by V_θ' and (5.6) by ℓ, add both expressions and integrate over $(0,\infty)$:

$$\lambda_0 \int_0^\infty \ell V_\theta'\, dx = \int_0^\infty (\ell V_\theta''' - \ell'' V_\theta')dx = \ell'(0)V_\theta'(0).$$

We know that $V'(0) > 0$ and also that $\ell'(0) \neq 0$ (otherwise ℓ should be identical zero), so $\ell'(0)V_\theta'(0) > 0$ and further we see that $\int_0^\infty \ell V_\theta'\, dx > 0$ and hence $\lambda_0 > 0$. □

6. PROOF OF THEOREM 1.1

Now we have gathered all pieces to prove Theorem 1.1. For the case $0 \le \theta < 1$ we put together Theorem 3.1 and Theorem 4.1. We examine the expression

(6.1) $$u(x,t) - U_{c(f)}(x-c(f)t-z_0) - V_\theta(x) + 1.$$

For $(x,t) \in Q^1\backslash\{(x,t) \mid c_1t < x \le c_1t + \delta\}$ we apply Theorem 4.1 and use the fact that $1 - V_\theta$ becomes exponentially small (see (1.18)) bounded by

$$1 - V_\theta(x) = O(e^{-\sqrt{-f'(1)}(c_1t+\delta)}), \quad t \to \infty$$

for this range of (x,t), so (6.1) becomes exponentially small. For $(x,t) \in Q^2 \cup \{(x,t) \mid c_1t \le x \le c_1t + \delta\}$ we apply Theorem 5.1 and use the fact that $1 - U_{c(f)}$ becomes exponentially small (see (1.5)) bounded by

$$1 - U_{c(f)}(x-c(f)t-z_0) < C_1e^{\beta_1((c_1-c(f))t+\delta-z_0)}$$

for this range of (x,t), so (6.2) becomes exponentially small, and finally we have proven (1.9).

There remains to look at the case $\theta = 1$. We can repeat section 5 with $\theta = 1$ and read then for $V_1(x) \equiv 1$. In this way we treat the strip $0 \le x \le \delta$, which we had to exclude from Q in Theorem 3.1. So (1.9) also is valid for $\theta = 1$. □

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